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CS109

Handout #32
May 9, 2016

Problem Set #5
Due: 2:30pm on Wednesday, May 18th

For each problem, **explain/justify how you obtained your answer** in order to obtain full credit. In fact, most of the credit for each problem will be given for the derivation/model used as opposed to the final answer. Make sure to describe the distribution and parameter values you used, where appropriate. It is fine for your answers to include summations, products, factorials, exponentials, or combinations, unless you are specifically asked for a computed numeric answer.

1. Consider the following function, which simulates repeatedly rolling a 6-sided die (where each integer value from 1 to 6 is equally likely to be "rolled") until a value ≥ 3 is "rolled".

```
int roll() {
   int total = 0;
   while (true) {                      // loop forever
      int roll = randomInteger(1, 6); // equally likely to return 1,...,6
      total += roll;
      if (roll >= 3) break;            // exit condition
   }
   return total;
}
```

   a. Let X = the value returned by the function `roll()`. What is E[X]?

   b. Let Y = the number of times that the die is "rolled" (i.e., the number of times that `randomInteger(1, 6)` is called) in the function `roll()`. What is E[Y]?

2. You go on a camping trip with two friends who each have a mobile phone. Since you are out in the wilderness, mobile phone reception isn't very good. One friend's phone will independently drop calls with 20% probability. Your other friend's phone will independently drop calls with 30% probability. Say you need to make 10 phone calls, so you randomly choose one of the two phones and you will use that same phone to make all your calls (but you don't know which has a 20% versus 30% chance of dropping calls). Of the first three (out of 10) calls you make, one of them is dropped. What is the conditional expected number of dropped calls in the 10 total calls you make (conditioned on having already had one of the first three calls dropped)?

3. Let X_1, X_2, … be a series of independent random variables which all have the same mean μ and the same variance σ^2. Let $Y_n = X_n + X_{n+1}$. For $j = 0, 1$, and 2, determine $\text{Cov}(Y_n, Y_{n+j})$. Note that you may have different cases for your answer depending on the value of j.

4. Let X_1, X_2, X_3, and X_4 be a set of pairwise uncorrelated random variables (i.e., $\rho(X_i, X_j) = 0$ when $i \neq j$), which all have the same mean μ and the same variance σ^2.

   a. What is the correlation $\rho(X_1 + X_2, X_3 + X_4)$?

   b. What is the correlation $\rho(X_1 + X_2, X_2 + X_3)$?

   c. What is the correlation $\rho(2X_1, X_1 + X_2)$?

5. Let's consider a simple dice game. Two fair 6-sided dice are rolled. Let X = the sum of the two dice. If X = 7, then the game ends and you win nothing (winnings = \$0). If $X \neq 7$, then you have the option of either stopping the game and receiving \$X (what you rolled on your last roll) or starting the whole process over again. Now consider this strategy to play: pick a number i, where $2 \leq i \leq 12$, and stop playing the first time that a value $\geq i$ is rolled (or until you are forced to stop as a result of rolling a 7). Define Y_i = winnings when you use this strategy with chosen value i. We are interested in the value i that maximizes the expected winnings $E[Y_i]$ over the all possible choices of i. To make a long story short, it turns out that the value of i that maximizes the expected winnings $E[Y_i]$ for the game is $i = 8$. For this problem, what we actually want is for you to *explicitly* compute the expected winnings $E[Y_i]$ for i = 5, 6, 8, and 9 to show why the expected winnings is maximized when $i = 8$. You do **not** need to consider the cases where i = 2, 3, 4, 10, 11, or 12.

6. Let X = the number of requests you receive at your web site per minute, where X ~ Poi(10). Each request, independently of all other requests, is equally likely to be routed to one of N web servers. Compute the expected number of web servers that will receive at least one request each during a minute. (Hint: there are a few ways to do this problem, but one way you might approach it is to first determine the *conditional* expectation of the number of web servers that receive at least one request each during a minute, conditioned on some fixed number, k, of requests during that minute. Then use that result to compute the *unconditional* expectation of the number of web servers that receive at least one request each during a minute.)

7. In class, we considered the following recursive function:

```
int Recurse() {
   int x = randomInteger(1, 3);  // equally likely to return 1, 2, or 3
   if (x == 1) return 3;
   else if (x == 2) return (5 + Recurse());
   else return (7 + Recurse());
}
```

   Let Y = the value returned by `Recurse()`. We previously computed E[Y] = 15. What is Var(Y)?

8. Program A will run 20 algorithms in sequence, with the running time for each algorithm being independent random variables with mean = 50 seconds and variance = 100 seconds^2. Program B will run 20 algorithms in sequence, with the running time for each algorithm being independent random variables with mean = 52 seconds and variance = 200 seconds^2.
   a. What is the approximate probability that Program A completes in less than 950 seconds?
   b. What is the approximate probability that Program B completes in less than 950 seconds?
   c. What is the approximate probability that Program A completes in less time than Program B?

9. A fair 6-sided die is repeatedly rolled until the total sum of all the rolls exceeds 300. Approximate the probability that *at least* 80 rolls are necessary to reach a sum that *exceeds* 300.

10. From past experience, we know that the midterm score for a student in CS106Z is a random variable with mean = 75. Assume that exam scores can be real values (i.e., fractional points can be given), but scores cannot be negative.
    a. Give an upper bound for the probability that a student's midterm score will be greater than or equal to 85.
    b. Now, say we are given the additional information that the variance of a student's midterm exam score in CS106Z is 25 (and you can use this information for parts (c) and (d) below as well). Give a bound on the probability that a student's midterm score is between 65 and 85, inclusive.
    c. According to Chebyshev's inequality, how many students would have to take the midterm in order to ensure, with at least 90% probability, that the class average would be within 5 of 75?
    d. According to the Central Limit Theorem, how many students would have to take the midterm in order to ensure, with at least 90% probability, that the class average would be within 5 of 75?

11. Say we have $X \sim \text{Poi}(20)$. We can directly compute $P(X \geq 25) \approx 0.1568$. Just for comparison, approximate $P(X \geq 25)$ using the Central Limit Theorem.

12. Open **`cs109.stanford.edu/problem12.html`** for the last question.

Yes, this problem set has only 12 problems on it. Be happy.